THE GENERAL TRANSFORMATION OF THE GROUP OF EUCLIDIAN MOVEMENTS.

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One of the most important of the transformation groups that occur in the Theory of Differential Equations, or in Geometry, is the so-called group of Euclidian Movements. We propose to investigate the *form* of the most general transformation of this group in the plane and in space.

— § 1. —

The infinitesimal transformations of the group of movements in the plane are $\left[\text{writing } p \text{ for } \frac{\partial f}{\partial x}, \text{ and } q \text{ for } \frac{\partial f}{\partial y}\right]$,

$$p\,, \quad q\,, \quad yp - xq\,,$$

the first two being the translations, and the third the rotation. The most general infinitesimal transformation of this G_3 (or three-fold group), has the form

$$Uf \equiv Ap + Bq + C\,(yp - xq)\,; \; A,\, B,\, C, \text{ consts.},$$

and since Uf and *const.* Uf may be considered as *equivalent* infinitesimal transformations, there are clearly ∞^2 infinitesimal transformations in all in this G_3.

We know that points which are absolutely invariant under the transformation

$$Uf \equiv (A + Cy)\,p + (B - Cx)\,q$$

are obtained by writing

$$A + Cy = 0\,, \quad B - Cx = 0\,, \qquad (C \neq 0)\,.$$

Hence, the only invariant point (within a finite distance of the origin) is

$$x = \frac{B}{C}\,, \quad y = -\frac{A}{C}\,.$$

In order to find the simplest form for the general infinitesimal transformation of the G_3 when the variables are referred to rectangular axes, let us transform the origin to the point $\frac{B}{C}\,,\ -\frac{A}{C}$ by introducing the new variables

$$x' = x - \frac{B}{C}\,, \quad y' = y + \frac{A}{C}\,.$$

We may assume $C \neq 0$, otherwise Uf would be a mere translation: hence, in the new variables Uf becomes

$$\begin{aligned} U'f &\equiv U(x')\frac{\partial f}{\partial x'} + U(y')\frac{\partial f}{\partial y'} \\ &\equiv U(x)\frac{\partial f}{\partial x'} + U(y)\frac{\partial f}{\partial y'} \\ &\equiv (A + Cy)\frac{\partial f}{\partial x'} + (B - Cx)\frac{\partial f}{\partial y'} \\ &\equiv C\left[y'\frac{\partial f}{\partial x'} - x'\frac{\partial f}{\partial y'}\right]; \end{aligned}$$

or as we may write it,

$$U'f \equiv y'\frac{\partial f}{\partial x'} - x'\frac{\partial f}{\partial y'}.$$

Hence we see that by a proper choice of the rectangular coordinate axes, *the most general transformation of the G_3 of movements in the plane may be written in the form of a mere rotation.*

The path-curves of $U'f$ are given by

$$\frac{dx'}{y'} = -\frac{dy'}{x'},$$

and hence they have the form

$$x'^2 + y'^2 = c^2; \qquad (c = \text{const.})$$

so that the path-curves of Uf are given by

$$\left[x - \frac{B}{C}\right]^2 + \left[y + \frac{A}{C}\right]^2 = c^2.$$

Since the finite equations of any transformation

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

are obtained by integrating the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dt,$$

it is clear that when the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

has been integrated, the finite equations of the infinitesimal transformation

may be obtained by a *quadrature*. Hence, the finite equations of the transformation

$$Uf \equiv (A + Cy)\frac{\partial f}{\partial x} + (B - Cx)\frac{\partial f}{\partial y}$$

may now be obtained by a quadrature.

If we indicate the *extended* transformation corresponding to Uf by

$$U^{(n)}f \equiv \xi(x, y)p + \eta(x, y)q + \eta'(x, y, y')q' + \ldots + \eta^{(n)}(x, y, y', y'' \ldots y^{(n)})q^{(n)}$$

where

$$y^{(i)} \equiv \frac{d^i y}{dx^i} \text{ and } q^{(i)} \equiv \frac{\partial^i f}{\partial y},$$

the *Differential Invariants* of the transformation Uf may be found by integrating the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta'} = \ldots = \frac{dy^{(n)}}{\eta^{(n)}}.$$

Prof. Lie has shown* that when we know the path-curves of the transformation, i. e. when the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

has been integrated, the other integral-functions of the simultaneous system may be found by one quadrature and a finite number of differentiations.

Thus we may consider the Differential Invariants of the transformation

$$Uf \equiv (A + By)p + (B - Cx)q$$

to be all known: and from this it follows that the whole Theory of Invariance of the G_3 of movements in the plane, including the theory of the congruence of plane curves, may be derived by operations which do not involve any further integration. This theory has been worked out in detail by Prof. Lie† in a different manner.

These results are very simple for the case considered above, but they are given here merely to indicate the manner in which such questions are to be treated.

* Compare "Page's Differential Equations," p. 69.

† "Continuirliche Gruppen," Chap. 22.

— § 2. —

The group of movements in space consists of the six infinitesimal transformations

$$p, \quad q, \quad r, \quad yp - xq, \quad zq - yr, \quad xr - zp,$$

the first three being the translations, and the other three the rotations.

The most general infinitesimal transformation of the G_6 has the form

$$Uf \equiv Ap + Bq + Cr + D(yp - xq) + E(zq - yr) + F(xr - zp),$$
$$A, B, \dots F \text{ consts.}$$

Since Uf and ρUf, where $\rho =$ const., are equivalent infinitesimal transformations, we may choose the above transformation in the form ρUf, where

$$\rho = \frac{1}{\sqrt{D^2 + E^2 + F^2}}.$$

It is clear that this is equivalent to considering the constants D, E, F to be connected by the relation

$$D^2 + E^2 + F^2 = 1: \tag{1}$$

and the reason for this assumption will appear from the sequel.

We may write Uf in the form

$$Uf \equiv (A + Dy - Fz)\,p + (B - Dx + Ez)\,q + (C - Ey + Fx)\,r:$$

and any invariant points, or curves consisting of absolutely invariant points, are obtained by writing

$$\left.\begin{aligned} A + Dy - Fz &= 0, \\ B - Dx + Ez &= 0, \\ C - Ey + Fx &= 0, \end{aligned}\right\}. \tag{2}$$

Multiplying these equations in order by E, F, D, and adding, we have

$$AE + BF + CD = 0. \tag{3}$$

Since $A, B, \dots F$ are undetermined constants, except for the relation (1), (3) will usually not be a true equation. We shall, however, divide our problem into two parts, by assuming, first, that (3) holds; and, secondly, that (3) does not hold.

I: We assume that the constants $A, B, \dots F$ have been chosen in such manner that

$$AE + BF + CD = 0. \tag{3}$$

It is then clear that the equations (2) are not independent ; and hence any two of them, say

$$\left.\begin{aligned} Fx - Ey + C &= 0 \\ Dx - Ez - B &= 0 \end{aligned}\right\} \tag{4}$$

determine a straight line which is invariant under Uf in such manner that each point on the line is separately invariant.

The line (4) cuts the xy-plane at the point

$$B/D\,,\ -A/D \qquad\qquad (D \neq 0)$$

and we choose this point as the origin by writing

$$x' = x - B/D\,, \quad y' = y + A/D\,, \quad z' = z\,.$$

The line (4) then becomes

$$\left.\begin{aligned} Fx' - Ey' &= 0 \\ Dx' - Ez' &= 0 \end{aligned}\right\} \tag{5}$$

while Uf assumes the form

$$\begin{aligned} U'f &\equiv U(x')\frac{\partial f}{\partial x'} + U(y')\frac{\partial f}{\partial y'} + U(z')\frac{\partial f}{\partial z'} \\ &\equiv U(x)\frac{\partial f}{\partial x'} + U(y)\frac{\partial f}{\partial y'} + U(z)\frac{\partial f}{\partial z'} \\ &\equiv (A + Dy - Fz)\frac{\partial f}{\partial x'} + (B - Dx + Ez)\frac{\partial f}{\partial y'} + (C - Ey + Fx)\frac{\partial f}{\partial z'} \\ &\equiv (Dy' - Fz')\frac{\partial f}{\partial x'} + (Ez' - Dx')\frac{\partial f}{\partial y'} + (Fx' - Ey')\frac{\partial f}{\partial z'}\,. \end{aligned}$$

This transformation is now made up of the three rotations ; and it was geometrically clear *a priori* that the three translations must disappear when the line (4) is invariant, since they leave no point within a finite distance of the origin invariant.

We shall now choose the invariant line (5) as the z-axis. This line is also represented by the equations

$$Fx' - Ey' = 0\,,$$

$$Fx' - Ey' - \frac{E^2 + F^2}{FD}(Dx' - Ez') = 0\,,$$

which may be written :

$$\left.\begin{aligned} &Fx' - Ey' = 0\,, \\ &EDx' + FDy' - (E^2 + F^2)z' = 0\,. \end{aligned}\right\} \tag{6}$$

The planes represented by (6) are now perpendicular; and a third plane through the origin perpendicular to these two is given by

$$Ex' + Fy' + Dz' = 0\,.$$

Let us now introduce new variables by writing

$$X \equiv Fx' - Ey'\,, \quad Y \equiv EDx' + FDy' - (E^2 + F^2)\,z'\,,$$

$$Z \equiv Ex + Fy' + Dz'\,.$$

We then have:

$$U'(X) \equiv Y, \quad U'(Y) \equiv -(E^2 + F^2 + D^2)X, \quad U'(Z) \equiv 0\,;$$

or, on account of (1),

$$U'f \equiv U(X)\frac{\partial f}{\partial x} + U(Y)\frac{\partial f}{\partial y} + U(Z)\frac{\partial f}{\partial z}$$

$$Y\frac{\partial f}{\partial x} - X\frac{\partial f}{\partial y}\,.$$

We see that this is an infinitesimal rotation in the variables X, Y, Z. Hence, *when the relation* (3) *exists, the most general movement possible in space is a rotation.* Geometrically, this was *a priori* evident; for when a straight line, consisting of invariant points, is invariant, it is clear that the only movement possible for the other points in space is a rotation around the invariant line.

II. Let us now suppose that no relation of the form

$$AE + BF + CD = 0$$

exists.

As in the last case, let us transform the origin to the point B/D, $-A/D$, by writing

$$x' = x - \frac{B}{D}, \quad y' = y + \frac{A}{D}, \quad z' = z\,. \qquad (D \neq 0)$$

The transformation Uf becomes

$$U'f \equiv (Dy' - Fz')\frac{\partial f}{\partial x'} + (Ez' - Dx')\frac{\partial f}{\partial y'} + (Fx' - Ey' + \varkappa)\frac{\partial f}{\partial z'}\,,$$

where

$$\varkappa \equiv \frac{AE + BF + DC}{D}$$

Now introduce the new independent variables X, Y, Z by means of:

$$X \equiv Fx' - Ey' + \varkappa\,, \quad Y \equiv EDx' + FDy' - (E^2 + F^2)z'\,,$$

$$Z \equiv Ex' + Fy' + Dz'\,;$$

and we find

$$U'f \equiv U(X)\frac{\partial f}{\partial x} + U(Y)\frac{\partial f}{\partial y} + U(Z)\frac{\partial f}{\partial z}$$

$$\equiv Y\frac{\partial f}{\partial x} - (X - \varkappa D^2)\frac{\partial f}{\partial y} + D\varkappa\frac{\partial f}{\partial z}.$$

Finally, introduce as new variables x_1, y_1, z_1, by means of

$$x_1 = X - \varkappa D^2, \quad y_1 = Y, \quad z_1 = Z/D\varkappa. \qquad (D \neq 0)$$

Hence the transformation assumes the form

$$U_1 f \equiv y_1\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial z_1}.$$

Hence, *the most general transformation of the group of movements in space*

$$Uf \equiv (A + Dy - Fz)p + (B - Dx + Ez)q + (C - Ey + Fx)r,$$

where the constants satisfy the relation

$$E^2 + F^2 + D^2 = 1,$$

may, by a proper choice of rectangular axes, be written in the canonical form

$$Uf \equiv yp - xq + r.$$

This transformation is evidently made up of a rotation around the z-axis, combined with a translation along that axis, i. e. all the points in space move on helixes wound upon right cylinders of which the z-axis is the axis. Hence, we have a result well known in kinematics that *the most general movement of a point in space is equivalent to a movement of that point on a helix.*

The aggregate of the transformations performed above upon Uf is clearly represented by

$$\left.\begin{aligned} x_1 &= F\left[x - \frac{B}{D}\right] - E\left[y + \frac{A}{D}\right] + \varkappa(1 - D^2), \\ y_1 &= ED\left[x - \frac{B}{D}\right] + FD\left[y + \frac{A}{D}\right] - (E^2 + F^2)z, \\ z_1 &= \left\{E\left[x - \frac{B}{D}\right] + F\left[y + \frac{A}{D}\right] + Dz\right\} \div D\varkappa, \end{aligned}\right\} \qquad (7)$$

and these equations enable us to pass from the first form of the infinitesimal transformation to the last at one step.

In the form

$$U_1 f \equiv y_1 p_1 - x_1 q_1 + r_1$$

it is obvious that the path-curves of the transformation are helixes; and they are given by the integrals of the simultaneous system:

$$\frac{dx_1}{y_1} = -\frac{dy_1}{x_1} = \frac{dz_1}{1},$$

in the form

$$x_1^2 + y_1^2 = c_1^2, \quad \tan^{-1}\frac{x_1}{y_1} - z_1 = c_2. \tag{8}$$

In order to obtain the path-curves of the transformation in its original form, we only need substitute in (8) the values of x_1, y_1, z_1, given by (7).

Since $U_1 f$ leaves the straight line $x_1 = y_1 = 0$ invariant, while the points on this line are interchanged among each other, it is clear that the original transformation always leaves the straight line

$$\left.\begin{array}{l} F(x - B/D) - E\left[y + \dfrac{A}{D}\right] + z(1 - D^2) = 0 \\ ED(x - B/D) + FD(y + A/D) - (E^2 + F^2)z = 0 \end{array}\right\}$$

similarly invariant:—and it is geometrically evident *a priori* that since the most general infinitesimal movement must always leave a straight line invariant, the most general movement to which a point of general position of space can be subjected is a movement on a helix.

It is clear that we may now obtain the finite equations of the original transformation by a quadrature. We know the *Invariants* (8) of the transformation; and if we *extend* the transformation under the hypothesis that x, y, and z are connected by a relation of the form

$$z = f(x, y)$$

we may, according to a theorem of Lie, find all the Differential Invariants of the G_6 of movements of the form

$$\Omega(x, y, z, p, q, r, s, t \ldots)$$

and establish the theory of the congruence of *surfaces* in space, *without further integration:* and a similar remark is true of *curves*, i. e. when x, y, z are connected by the relations

$$z = \varphi(x), \quad y = \psi(x).$$

These theories have been developed by Lie, in detail, from another standpoint.